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An exact analysis of spontaneous emission by a single two level atom in the rotating wave approximation I. Analytic results

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Abstract. The problem of a single two level atom in its excited state, placed in a cavity in which no photons are present initially, and interacting with N modes of the electromagnetic field, is solved exactly in the electric dipole, rotating wave approximation. Expressions for the energy eigenvalues and eigenvectors are derived, and an exact representation for the time dependent wavefunction of the system is given. These quantities are evaluated analytically for $N = 1, 2$ and 3, and the time dependent properties of the system for these values of N are investigated and discussed. In the following paper results for larger values of N which have been found numerically are presented.

1. Introduction

The system we consider in this paper is that of a single, two level atom which is coupled to N modes of the electromagnetic field. The interaction between the atom and the field is assumed to be electric dipole, and the rotating wave approximation is made, but within the framework of these approximations the model is solved exactly. The initial state of the system is that in which the atom is in its excited state and no photons are present, so that the atom will begin to decay by spontaneous emission. We find the exact energy eigenvalues and eigenvectors of the Hamiltonian, and use a linear combination of the latter (with time dependent coefficients) to represent the time dependent wavevector of the system. We thus obtain an exact expression for the state vector which can be used to calculate the time dependent properties of the system, such as the probability that a photon will be present at a certain time.

This model, although obviously a simplified one, is of some importance in quantum optics. Most quantum mechanical theories of the laser, for example, are essentially one atom theories (eg Lamb and Scully 1967)—the results for N_A atoms simply being taken to be N_A times the results for one atom (Fleck 1966). Thus exact treatments of this model are of considerable interest, not only for the results which can be derived from them, but also as a yardstick in judging the validity of more approximate calculations. For these reasons we analyse this model in detail.

Previous work in finding exact solutions to the eigenvalue problem was performed by Jaynes and Cummings (1963) who considered a single atom interacting with a single field mode. Fleck (1966) gives a more detailed account of their solution. Tavis and Cummings (1968) presented exact results for the Hamiltonian in which N_A atoms interact

with one mode of the field at resonance (ie when all the atoms have identical energies and this energy is equal to that of the field mode). Walls and Barakat (1970) and Scharf (1970) independently found solutions to this problem. Mallory (1969) has treated the more general case in which the field mode and the atoms need not be in resonance. The multimode case has been treated by Swain (1972a) who found solutions to the problem in which N modes of the field (with arbitrary frequencies) interact with N_A identical atoms. Walls (1971) has applied somewhat similar methods to problems in nonlinear optics. Mallory and Scharf use methods based on the Bargmann representation for the field states, whilst the other authors quoted work with the number representation.

In all the treatments of the N_A atom problem listed so far the atoms were assumed to be situated completely within a volume small compared to the wavelength of the field mode(s) so that they could be represented by a set of pseudospin operators of magnitude $\frac{1}{2}N_A$. (In addition, the use of the pseudospin representation requires that all the atoms have the same energy.) Swain (1972b) has been able to relax these conditions and so give a solution to the problem of N field modes interacting with N_A atoms which need not be localized in space nor have identical energy separations.

The problem we consider is related to the more general problem of spontaneous emission by an assembly of atoms which was first studied by Dicke (1954). He found that under certain circumstances the radiation rate is proportional to the square of the number of atoms—a phenomenon which is known as super-radiance. Many authors have further considered this problem, both in the case when the atoms are confined to a region small compared with the mode wavelength and when they are distributed over a large volume (see, for example, Bonifacio and Preparata 1970, Dialetis 1970, Eberly and Rehler 1971, and Arecchi and Courtens 1970). (This is not intended to be an extensive bibliography.) We shall give a numerical analysis of the Dicke problem in a later publication.

The closest approach to our work is to be found in the series of papers by Davidson and Kozak. In the first of these (Davidson and Kozak 1970a) they use a master equation approach to investigate the decay of a single, initially excited atom which interacts with all the modes of the electromagnetic field. (The limit as the volume of the system tends to infinity is taken.) In the second paper (Davidson and Kozak 1970b) they consider the case when the volume is finite, which leads to consideration of Poincaré recurrences. Their solution has some unphysical properties in that they find that the probabilities they calculate can sometimes be negative. This point is taken up in the third paper (Davidson and Kozak 1971) where they also find an exact expression for the probability of one atom interacting with all the modes of the electromagnetic field being in its excited state at time t.

Our treatment differs from theirs in that we use a different approach and give a more general (and more complete) discussion of the problem. Thus we calculate the exact eigenvectors, eigenvalues and time dependent wavefunction of the system whereas they concentrate on the calculation of probability amplitudes. In the main we restrict our attention to the calculation of the properties of a system with a finite number of field modes (although our equations are applicable to the case of an infinite number of modes) whereas they are concerned exclusively with the infinite number of modes problem. Their main objective is a comparison of quantum mechanical and quantum statistical approaches to the same problem.

In this paper we derive the fundamental formulae and give exact analytic results in the cases $N = 1, 2$, and 3 for particular energy level schemes. In the following paper we give numerical evaluations of the expressions derived here for larger values of N.

In § 2 we describe the model and discuss briefly the rotating wave approximation. Exact eigensolutions appropriate to the problem of spontaneous emission by a single atom are derived in § 3, and an exact representation for the time dependent wavefunction is obtained in § 4. In § 5 analytic solutions to the cases in which one, two and three modes interact with the atom are given.

2. The model Hamiltonian

The Hamiltonian for a single two level atom interacting with N modes of the electromagnetic field in the electric dipole approximation is

$$H = \sum_{\lambda=1}^{N} (a_\lambda^\dagger a_\lambda + \tfrac{1}{2})\omega_\lambda + (\sigma^3 + \tfrac{1}{2})\omega_0 + \sum_{\lambda=1}^{N} (g_\lambda a_\lambda \sigma^+ + g_\lambda^* a_\lambda^\dagger \sigma^-) + \sum_{\lambda=1}^{N} (g_\lambda a_\lambda \sigma^- + g_\lambda^* a_\lambda^\dagger \sigma^+) \tag{1}$$

(we use a system of units in which $\hbar = 1$). Here $a_\lambda^\dagger$ creates a photon in the mode λ whose frequency is ω_λ and wavevector $\boldsymbol{k}_\lambda$. The operators σ^+, σ^- and σ^3 are spin operators and are used here to describe the two level atom. They are defined by their action on the energy eigenstates of the unperturbed atom, which are $|\frac{1}{2}\rangle$ and $|-\frac{1}{2}\rangle$. Thus

$$\sigma^3|\alpha\rangle = \alpha|\alpha\rangle \qquad \alpha = \pm\tfrac{1}{2} \tag{2}$$

and

$$\sigma^+|\alpha\rangle = \delta(\alpha+\tfrac{1}{2})|+\tfrac{1}{2}\rangle \qquad \sigma^-|\alpha\rangle = \delta(\alpha-\tfrac{1}{2})|-\tfrac{1}{2}\rangle \tag{3}$$

where $\delta(\alpha - x) = 1$ if $\alpha = x$, and is zero otherwise. ω_0 is the energy separation of the two atomic levels. g_λ is the coupling constant and is given explicitly as

$$g_\lambda = -\mathrm{i}(2\pi\omega_\lambda V)^{-1/2}\boldsymbol{d}\cdot\boldsymbol{u}_\lambda \tag{4}$$

where V is the volume of the system, $\boldsymbol{d}$ is the dipole matrix element, and $\boldsymbol{u}_\lambda$ is the λth normal mode function for the cavity evaluated at the position of the atom.

The final term in (1) contains the so called antiresonant contributions, and it is almost universal in calculations in quantum optics to neglect this term. (This is the rotating wave approximation (RWA).) Although there is no *a priori* justification for neglecting the antiresonant terms, it seems reasonable to assume that their inclusion would produce quantitative differences in the curves obtained using the RWA, but no significant qualitative changes, at least if the coupling constant is sufficiently small. In a perturbation treatment, the terms neglected would always be small compared to the ones retained, and their omission presumably implies that certain small shifts and high frequency modulations have been omitted. Thus the rotating wave approximation is expected to provide a realistic description of the behaviour of the system. However, caution must be exercised in the use of the RWA because the full interaction connects more states than does the RWA. For example, if we consider a system with only one atom and one field mode with an initial state $|-\frac{1}{2}\rangle|0\rangle$ corresponding to the atom being in its ground state with no photons present, then in the RWA this state is connected to no other state, whereas with the full interaction, the state $|+\frac{1}{2}\rangle|1\rangle$ is connected. Thus if one is considering a problem in which such transitions are important the RWA could lead to incorrect conclusions.

It is also convenient to change the zero of energy so that the zero point energy $\Sigma_\lambda \frac{1}{2}\omega_\lambda$ of the electromagnetic field and the term $\frac{1}{2}\omega_0$ in (1) may be set to zero. When these measures are taken the Hamiltonian becomes

$$H = \sum_{\lambda=1}^{N} a_\lambda^\dagger a_\lambda \omega_\lambda + \sigma^3 \omega_0 + \sum_{\lambda=1}^{N} (g_\lambda a_\lambda \sigma^+ + g_\lambda^* a_\lambda^\dagger \sigma^-). \tag{5}$$

We present an exact study of this Hamiltonian in the subsequent sections.

3. The eigensolutions

It has been shown (Swain 1972a) that the exact eigenvectors of (5) are of the form

$$|q, c\rangle = \sum_{\boldsymbol{n}=0}^{\infty} \sum_{\alpha=\pm 1/2} A_{q,c}(\boldsymbol{n}, \alpha) \delta\left(c - \sum_\lambda n_\lambda - \alpha\right) |\boldsymbol{n}\rangle |\alpha\rangle \tag{6}$$

where c is an eigenvalue of the operator

$$C = \left(\sum_\lambda a_\lambda^\dagger a_\lambda + \sigma^3\right) \tag{7}$$

and q is an eigenvalue of the operator

$$Q = \sum_\lambda (a_\lambda^\dagger a_\lambda \omega_{\lambda,0} + g_\lambda a_\lambda \sigma^+ + g_\lambda^* a_\lambda^\dagger \sigma^-) \qquad \omega_{\lambda,0} = \omega_\lambda - \omega_0. \tag{8}$$

It is clear that

$$H = C\omega_0 + Q. \tag{9}$$

From (7), the allowed values of c are evidently $c = -\frac{1}{2}, +\frac{1}{2}, 1\frac{1}{2} \ldots$. The vector $\boldsymbol{n}$ labels the set of non-negative integers $\boldsymbol{n} = (n_1, n_2, \ldots, n_N)$ and $\Sigma_{\boldsymbol{n}=0}^{\infty} \equiv \Sigma_{n_1=0}^{\infty} \Sigma_{n_2=0}^{\infty} \cdots \Sigma_{n_N=0}^{\infty}$. The numerical coefficients $A_{q,c}(\boldsymbol{n}, \alpha)$ must satisfy the relationship

$$\left(\sum_\lambda n_\lambda \omega_{\lambda,0} - q\right) A_{q,c}(\boldsymbol{n}, \alpha) + \sum_\lambda \{g_\lambda (n_\lambda + 1)^{1/2} \delta(\alpha - \tfrac{1}{2}) A_{q,c}(\boldsymbol{n}', n_\lambda + 1, -\tfrac{1}{2}) + g_\lambda^* (n_\lambda)^{1/2} \delta(\alpha + \tfrac{1}{2}) A_{q,c}(\boldsymbol{n}', n_\lambda - 1, +\tfrac{1}{2})\} = 0 \tag{10}$$

for every set $\boldsymbol{n}, \alpha$ which satisfies the relation

$$c = \sum_\lambda n_\lambda + \alpha. \tag{11}$$

Here $\boldsymbol{n}', n_\lambda + 1 \equiv n_1, n_2, \ldots n_{\lambda-1}, n_\lambda + 1, n_{\lambda+1}, \ldots n_N$. $|\boldsymbol{n}\rangle$ is an eigenvector of the number operator $\Sigma_\lambda a_\lambda^\dagger a_\lambda$.

Equations (10) and (11) clearly define a set of linear inhomogeneous equations; the condition that the set have consistent solutions leads to an eigenvalue equation for q. When q has been found, it is clear from (9) that the eigenvalues of H corresponding to the eigenkets (6) are $c\omega_0 + q$:

$$H|q, c\rangle = (c\omega_0 + q)|q, c\rangle. \tag{12}$$

We are now in a position to determine the eigenvalues and eigenvectors appropriate to a single atom initially placed in the cavity in an excited state with no photons present.

The initial state vector is

$$|\psi(0)\rangle = |0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle. \tag{13}$$

$|\psi(0)\rangle$ as defined by (13) is an eigenvector of C belonging to the eigenvalue $c_0 = \frac{1}{2}$ (corresponding to $\boldsymbol{n} = 0$, $\alpha = +\frac{1}{2}$). We note that C commutes with H, and thus if the system is initially prepared in an eigenstate of C having eigenvalue c, it will remain in an eigenstate of C belonging to the same eigenvalue c for all time. Thus a complete set of states for describing a single atom placed in its excited state in a cavity at $t = 0$ with no photons present initially is obtained by listing all the eigenstates of C belonging to the eigenvalue c_0, that is, the set $|\boldsymbol{n}\rangle|\alpha\rangle$ for all values of $\boldsymbol{n}$ and α such that

$$\sum_{\lambda} n_\lambda + \alpha = \tfrac{1}{2} \tag{14}$$

is satisfied.

Apart from $\boldsymbol{n} = 0$, $\alpha = \frac{1}{2}$, the only other sets of $\boldsymbol{n}$, α which satisfy (14) are

$$n_\lambda = \begin{cases} 0 & \lambda \neq K \\ 1 & \lambda = K \end{cases} \qquad \alpha = -\tfrac{1}{2} \text{ for } K = 1, 2, \ldots, N. \tag{15}$$

The complete set of equations (10) for the given value of $c = c_0$ is then

$$\boldsymbol{n} = 0, \alpha = +\tfrac{1}{2}: \qquad -qB_q(0) + \sum_{\lambda} g_\lambda B_q(\lambda) = 0 \tag{16}$$

$$\boldsymbol{n}' = 0, n_K = 1, \alpha = -\tfrac{1}{2}: \qquad (\omega_{K0} - q)B_q(K) + g_K^* B_q(0) = 0 \tag{17}$$

where

$$B_q(0) \equiv A_{1/2,q}(0, 0, \ldots, 0, +\tfrac{1}{2}) \tag{18}$$

and

$$B_q(K) \equiv A_{1/2,q}(0, 0, \ldots, 1_K, \ldots, 0, -\tfrac{1}{2}). \tag{19}$$

The system of equations (16) and (17) has a consistent solution if

$$q - \sum_{\lambda} \frac{|g_\lambda|^2}{q - \omega_{\lambda,0}} = 0. \tag{20}$$

If this is solved for the q, then the $B_q(\lambda)$ are given in terms of $B_q(0)$ by

$$B_q(\lambda) = \frac{g_\lambda^* B_q(0)}{q - \omega_{\lambda,0}} \tag{21}$$

where henceforth q is to be interpreted as one of the solutions of (20). Consequently the set of eigenvectors of (5) having $c = \frac{1}{2}$ is

$$|\tfrac{1}{2}, q\rangle = B_q(0)\left(|0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle + \sum_{\lambda} \frac{g_\lambda^*}{q - \omega_{\lambda,0}} |0, 0, \ldots, 1_\lambda, \ldots, 0\rangle|-\tfrac{1}{2}\rangle\right) \tag{22}$$

and these belong to the eigenvalue of (5)

$$E(\tfrac{1}{2}, q) = (\tfrac{1}{2}\omega_0 + q). \tag{23}$$

By requiring that (22) be normalized we find

$$|B_q(0)|^2 = \left(1 + \sum_{\lambda} \frac{|g_\lambda|^2}{(q - \omega_{\lambda,0})^2}\right)^{-1}. \tag{24}$$

Thus (24) determines $B_q(0)$ apart from an arbitrary phase factor, which is of no physical significance, as all the physical properties calculated depend upon $|B_q(0)|^2$, and not $B_q(0)$. The eigenvectors (22) with $B_q(0)$ given by (24) are orthonormal:

$$\langle \tfrac{1}{2}, q | \tfrac{1}{2}, q' \rangle = \delta(q-q'). \tag{25}$$

Equations (21)–(25) give exact expressions for the eigenvectors and eigenvalues of the Hamiltonian (5) appropriate to spontaneous emission by an isolated atom.

4. Time dependent properties

We are interested in calculating the evolution of the system from the particular initial state $|0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle$ corresponding to no initial excitation of the field and the atom in its excited state. Since this ket is an eigenket of C (belonging to $c = \frac{1}{2}$) we can expand the state of the system at time t in terms of the kets $|\frac{1}{2}, q\rangle$, that is, we may write

$$|\psi(t)\rangle = \sum_q \beta(q, t)|\tfrac{1}{2}, q\rangle \tag{26}$$

subject to

$$|\psi(0)\rangle = |0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle. \tag{27}$$

The sum over q extends over all the allowed values of q as determined by (20). If we substitute from (26) into the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = H|\psi(t)\rangle \tag{28}$$

we can solve for $\beta(q, t)$. Since $|\frac{1}{2}, q\rangle$ is an eigenstate of H belonging to the eigenvalue (23) we easily find that

$$\beta(q, t) = \exp\{-\mathrm{i}(\tfrac{1}{2}\omega_0 + q)t\}\beta(q, 0). \tag{29}$$

To evaluate $\beta(q, 0)$ we make use of the fact that $|\psi(0)\rangle$ is given by (27):

$$|\psi(0)\rangle = \sum_q \beta(q, 0)|\tfrac{1}{2}, q\rangle \equiv |0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle. \tag{30}$$

Taking the scalar product of (30) with $|\frac{1}{2}, q'\rangle$ and making use of (22), and (25), we find

$$\begin{aligned}\langle \tfrac{1}{2}, q'|\psi(0)\rangle = \beta(q', 0) &= \langle \tfrac{1}{2}, q'|0, 0, \ldots, 0\rangle|+\tfrac{1}{2}\rangle \\ &= B_{q'}^*(0)\end{aligned} \tag{31}$$

that is

$$\beta(q, 0) = B_q^*(0). \tag{32}$$

Combining (26), (29) and (32), we find

$$|\psi(t)\rangle = \exp(-\tfrac{1}{2}\mathrm{i}\omega_0 t)\sum_q B_q^*(0)\exp(-\mathrm{i}qt)|\tfrac{1}{2}, q\rangle \tag{33}$$

for the exact time dependent state vector for the system. ($B_q^*(0)$ is given by (24).)

Once this ket is known we can calculate the time dependent properties of interest. For example, the probability that there is a photon in the mode λ at time t is given by

$$p_\lambda(t) = \sum_{\alpha = \pm 1/2} |\langle \alpha|\langle 0, 0, \ldots, 1_\lambda, \ldots, 0|\psi(t)\rangle|^2. \tag{34}$$

On using (33) and (22) in (34), we find that the only value of α which contributes is $\alpha = -\frac{1}{2}$, and (34) reduces to

$$p_\lambda(t) = \left| \sum_q \frac{g_\lambda^* |B_q(0)|^2 \, e^{-iqt}}{q - \omega_{\lambda,0}} \right|^2 . \tag{35}$$

Similarly, the probability $P_0(t)$ that there are no photons in the field at time t is given by

$$P_0(t) = \sum_{\alpha = \pm 1/2} |\langle \alpha | \langle 0, 0, \ldots, 0 | \psi(t) \rangle|^2 \tag{36}$$

$$= \left| \sum_q |B_q(0)|^2 \, e^{-iqt} \right|^2 . \tag{37}$$

$P_0(t)$ may also be interpreted as the probability that the atom will be in its excited state at time t.

In the numerical work which follows we have concentrated in the main on calculating $P_0(t)$ because, apart from its own importance, it is simply related to other properties of interest. For example, in the case of photon counting distributions (Pike 1969) the quantity one wishes to calculate is $\rho_n(t)$, the probability that there are n photons in the field at time t. Now for the problem we are considering, the choice of our initial state and the fact that C as given by (7) is a constant of the motion implies that there can only be on average one or no photons in the field at any time, and so

$$\rho_0(t) = P_0(t), \qquad \rho_1(t) = 1 - P_0(t). \tag{38}$$

Thus in this case the photon counting distribution is very simple. (See Bonifacio and Preparata (1970) for the more interesting calculation of the photon counting distributions for the case of N_A atoms interacting with one mode on resonance.)

Also the mean number of photons present in the field at time t, $\bar{n}(t)$, is given by

$$\bar{n}(t) = \sum_{n=0}^{\infty} n \rho_n(t) \tag{39}$$

$$= 1 - P_0(t) \tag{40}$$

on using (38).

5. Analytic results

In this section we derive exact analytic expressions for the cases in which the atom interacts with one, two, and three light modes respectively. The behaviour of the system in these simple cases has many features in common with the behaviour in the many mode case (see paper II). The energy level schemes for the three cases are shown in figure 1. We have shown these energy level schemes because the analytic forms of the expressions obtained are particularly simple in these cases. It is straight forward to solve (20) for the eigenvalues q using elementary (but sometimes lengthy) algebra, and hence calculate the other properties of the system. We do not give details of the calculation but present only the results.

5.1. One mode

This problem has been treated by Jaynes and Cummings (1963) and Fleck (1966), but it is of interest to discuss the results here so as to compare them with more complicated

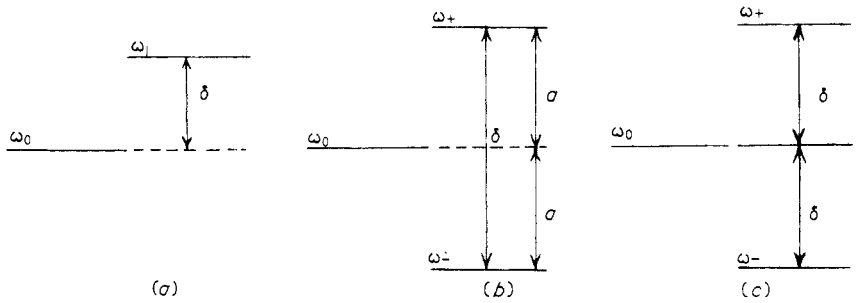

Figure 1. The unperturbed energy level schemes for (*a*) one, (*b*) two and (*c*) three modes.

cases. From (20), the eigenvalues q appropriate to the energy scheme for one mode given in figure 1 are

$$q_{\pm} = \tfrac{1}{2}\{\delta \pm (\delta^2 + 4|g_\lambda|^2)^{1/2}\}. \tag{41}$$

Having regard to equation (23) which specifies the eigenvalues of H we see that (41) implies that the energy levels of the interacting system are placed symmetrically about the average energy of the atom and the mode for the noninteracting system. We note that if $\delta^2 \ll 4|g_\lambda|^2$ then the energy eigenvalues (belonging to $c = \frac{1}{2}$) for the perturbed system, which are approximately $\frac{1}{2}\omega_0 \pm |g|$, are very different from those of the unperturbed system (also belonging to $c = \frac{1}{2}$) which are $\frac{1}{2}\omega_0, \frac{1}{2}\omega_0 + \delta$. The above condition corresponds to the limit of very strong interaction between the atom and the field, and in this limit the eigenkets $|0\rangle|+\frac{1}{2}\rangle$ and $|1\rangle|-\frac{1}{2}\rangle$ of the unperturbed system contribute equally to the eigenkets $|\frac{1}{2}, q\rangle$ of the perturbed system given by (22).

Substituting from (41) into (21) and (37) for $B_q(0)$ and $P_0(t)$ we find that

$$|B_{\pm q}(0)|^2 = \frac{(\delta^2 + 4|g_\lambda|^2)^{1/2} \mp \delta}{2(\delta^2 + 4|g_\lambda|^2)^{1/2}} \tag{42}$$

and

$$P_0(t) = 1 - \frac{4|g_\lambda|^2}{\delta^2 + 4|g_\lambda|^2} \sin^2\{\tfrac{1}{2}(\delta^2 + 4|g_\lambda|^2)^{1/2} t\}. \tag{43}$$

We note that (43) implies that only if $\delta = 0$ does $P_0(t)$ oscillate sinusoidally between 0 and 1. For $|\delta| > 0$, the minimum of $P_0(t)$ is $\delta^2/(\delta^2 + 4|g_\lambda|^2)$ so that the probability of the atom being in its ground state is never unity. When $|\delta|$ is very large ($\delta^2 \gg 4|g_\lambda|^2$) it is clear that $1 \geqslant P_0(t) \geqslant 1 - 4|g_\lambda|^2/\delta^2$ so that $P_0(t)$ deviates only slightly from the value one. This is consistent with the interpretation that when the atom and the field have very different energies the effective interaction between them is small.

5.2. *Two modes*

For the two and three mode situations it is necessary to make the assumption that $|g_\lambda|^2$ is independent of the mode λ in order to obtain manageable algebraic expressions. This will normally be a reasonable approximation providing that the total separation in energy of the field modes is small (ie if $|\delta| \ll \omega_0$ in the two mode case), but it is an approximation often made when this is not so (eg Davidson and Kozak 1970a) particularly when dealing with an infinite number of modes. The exact functional form

of g_λ depends on the shape of the cavity through the factor u_λ in (4), and rather than assume an explicit functional form for g_λ appropriate to a particular cavity it is preferred to take the simplest case, g_λ = constant.

If one makes this assumption one finds for the allowed values of q in the two mode case

$$q_0 = 0, \qquad q_\pm = \pm(a^2+2|g|^2)^{1/2}, \qquad a = \tfrac{1}{2}\delta. \tag{44}$$

By referring to (23), it is apparent that if $a^2 \ll 2|g|^2$ (effectively a very strong interaction) the energy eigenvalues in the perturbed system differ considerably from those in the unperturbed system. On the other hand, if $a^2 \gg 2|g|^2$ (very weak interaction) the eigenvalues are approximately the same in both the perturbed and unperturbed systems. This is analogous to the one mode case.

After some algebra one obtains the following expressions for the $|B_q(0)|$:

$$|B_{q_0}(0)|^2 = \frac{a^2}{a^2+2|g|^2}, \qquad |B_{q\pm}(0)|^2 = \frac{|g|^2}{a^2+2|g|^2}. \tag{45}$$

Using (44) and (45) in (37) we find

$$P_0(t) = \left(\frac{a^2+2|g|^2\cos(\theta t)}{\theta^2}\right)^2 \tag{46}$$

where

$$\theta = (a^2+2|g|^2)^{1/2}. \tag{47}$$

$P_0(t)$ is equal to unity whenever

$$t = n\pi/\theta, \qquad n = 0, 1, 2, \ldots, \tag{48}$$

but it is apparent from (46) that $P_0(t)$ is never zero if the condition $a^2 > 2|g|^2$ is satisfied. Hence if the mode spacing is sufficiently large the probability of the atom being in its ground state is never unity.

For the two mode case it is also of interest to calculate $p_+(t)$ and $p_-(t)$, the probabilities that a photon will be present at time t in modes with energies ω_0+a and ω_0-a respectively. From (35), (44) and (45) we find, after some algebraic simplification, that

$$p_+(t) = p_-(t) = \frac{2|g|^2}{\theta^4}[a^2\{1-\cos(\theta t)\}+|g|^2\sin^2(\theta t)]. \tag{49}$$

This function is zero whenever $\cos(\theta t) = 1$, that is whenever (48) is satisfied.

In the strong interaction limit, $a^2 \ll 2|g|^2$, (46) and (49) reduce to

$$\left.\begin{aligned} P_0(t) &\sim \cos^2(\sqrt{2}|g|t) \\ p_\pm(t) &\sim \tfrac{1}{2}\sin^2(\sqrt{2}|g|t) \end{aligned}\right\} \qquad a^2 \ll 2|g|^2 \tag{50}$$

whilst in the weak interaction case, $a^2 \gg 2|g|^2$, they give

$$\left.\begin{aligned} P_0(t) &\sim 1+\frac{4|g|^2}{a^2}\{\cos(at)-1\} \\ p_\pm(t) &\sim \frac{2|g|^2}{a^2}\{1-\cos(at)\} \end{aligned}\right\} \qquad a^2 \gg 2|g|^2. \tag{51}$$

Thus in both the strong and weak interaction limits sinusoidal behaviour is obtained. Somewhat similar behaviour also occurs in the many mode cases.

In figure 2 is plotted $P_0(t)$ for the intermediate case $a^2 = |g|^2$ as a function of the dimensionless 'time' $|g|t$. It is seen to be a periodic function with alternatively high ($P_0(t) = 1$) and low ($P_0(t) \sim 0{\cdot}61$) peaks. As the ratio $a^2/|g|^2$ is decreased (increasing interaction) the height of the lower peak gradually increases, approaching the value unity as $a^2/|g|^2$ tends to zero. This is the limit (50). As the ratio $a^2/|g|^2$ is increased (decreasing interaction) the height of the lower peak gradually decreases, until beyond the value 2, $P_0(t)$ is never zero, and as $a^2/|g|^2$ increases indefinitely the limit (51) is obtained.

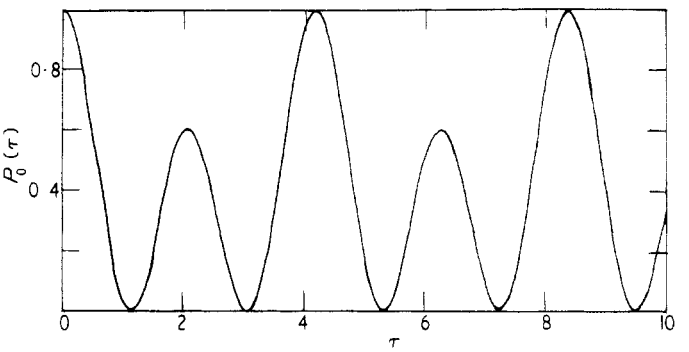

Figure 2. P_0 as a function of τ in the two mode case for $a^2 = |g|^2$.

We emphasize that the behaviour of $P_0(t)$ and $p_\pm(t)$ in the two mode case described here is always periodic; this is to be contrasted with the three mode case.

A useful measure of the extent to which the energy is distributed over the field modes and the atom is provided by the following quantities, $\bar{P}_0, \bar{p}_+, \bar{p}_-$ which are defined by the relation

$$\bar{p} = \lim_{T\to\infty} \frac{1}{T}\int_0^T p(t)\,\mathrm{d}t. \tag{52}$$

They may be calculated from equations (46) and (49) by noting that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \cos^2(qt)\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{T}\int_0^T \sin^2(qt)\,\mathrm{d}t = \frac{1}{2}$$

whilst the integrals of $\cos(qt)$ and $\sin(qt)$ alone vanish. Hence we find

$$\bar{P}_0 = \frac{a^4+2|g|^4}{(a^2+2|g|^2)^2} \tag{53}$$

and

$$\bar{p}_\pm = \frac{|g|^2(2a^2+|g|^2)}{(a^2+2|g|^2)^2}. \tag{54}$$

(53) and (54) give the average distribution of the total energy of the system over its constituent parts—the atom and the two field modes—as a function of a^2 and $|g|^2$. (Actually, (53) and (54) depend only on the ratio $a^2/|g|^2$.) Note that as the limit $a^2 \gg 2|g|^2$

is obtained, $\bar{P}_0 \to 1-4|g|^2/a^2$, $\bar{p}_\pm \to 2|g|^2/a^2$ so that the energy tends to reside entirely in the atom in the weak interaction limit, whilst in the opposite limit, $a^2 \ll 2|g|^2$, we find $\bar{P}_0 \to \frac{1}{2}$, $\bar{p}_\pm \to \frac{1}{4}$ so that the energy is divided in the first place equally between the atom and the field and in the second place, equally between the two field modes. For the case plotted in figure 2, $a^2 = |g|^2$, $\bar{P}_0 = \frac{1}{3}$, and $\bar{p}_\pm = \frac{1}{3}$.

5.3. Three modes

In this case, (20) has four solutions for q corresponding to the four eigenvalues of H (equation (23)). These are $\pm q_+$, $\pm q_-$ where

$$q_\pm = \{\tfrac{1}{2}(\delta^2 + 3|g|^2 \pm \sqrt{x})\}^{1/2} \tag{55}$$

and

$$x = \delta^4 + 2\delta^2|g|^2 + 9|g|^4. \tag{56}$$

We note that in the one, two and three mode cases the energy eigenvalues for the interacting system are symmetric about the mean of the unperturbed energies of the atom and the modes.

$B_q(0)$ is found to be an even function of q:

$$|B_{\pm q_\pm}(0)|^2 = |B_{q_\pm}(0)|^2 = \frac{\sqrt{x} \mp (\delta^2 - 3|g|^2)}{4\sqrt{x}}. \tag{57}$$

Using (55) and (57) in (37) and (35) we obtain the following expressions for $P_0(t)$ and $p_\pm(t)$:

$$P_0(t) = \frac{1}{x}\{r_+ \cos(q_+ t) + r_- \cos(q_- t)\}^2 \tag{58}$$

$$p_0(t) = \frac{|g|^2}{x}\left(\frac{r_+ \sin(q_+ t)}{q_+} + \frac{r_- \sin(q_- t)}{q_-}\right)^2 \tag{59}$$

$$p_+(t) = p_-(t) = \frac{|g|^2}{x}[\delta^2\{\cos(q_+ t) - \cos(q_- t)\}^2 + \{q_+ \sin(q_+ t) - q_- \sin(q_- t)\}^2] \tag{60}$$

where

$$r_\pm = \tfrac{1}{2}\{\sqrt{x} \pm (3|g|^2 - \delta^2)\}. \tag{61}$$

In contrast with the one and two mode cases, $P_0(t)$ can take the value zero (for infinitely many values of t) whatever the relative magnitudes of $|g|$ and δ. It is reasonable to associate this property with the fact that we always have a mode on resonance with the atom.

In further contrast with the one and two mode situations the functions defined in equations (53), (54) and (55) are no longer periodic functions, but belong to the class of 'almost periodic functions' (Bohr 1947). If we take $P_0(t)$ as an example, this implies that $P_0(t)$ has the value unity at $t = 0$ and at *no other time*. However, it will take a value arbitrarily close to unity (say 0·99) an infinite number of times. This is connected with the problem of Poincaré recurrences (Bocchieri and Loinger 1957, Percival 1961). Thus if $\tau(\epsilon)$ is the mean time interval between occasions when $P_0(t)$ has the value $P_0(t) = 1-\epsilon$, $\tau(\epsilon)$ is the Poincaré recurrence time.

In figure 3 we have plotted $P_0(t)$, $p_0(t)$, and $p_\pm(t)$ as functions of the dimensionless 'time' $|g|t$ for the case where $\delta = |g|$. Comparison with the periodic behaviour in

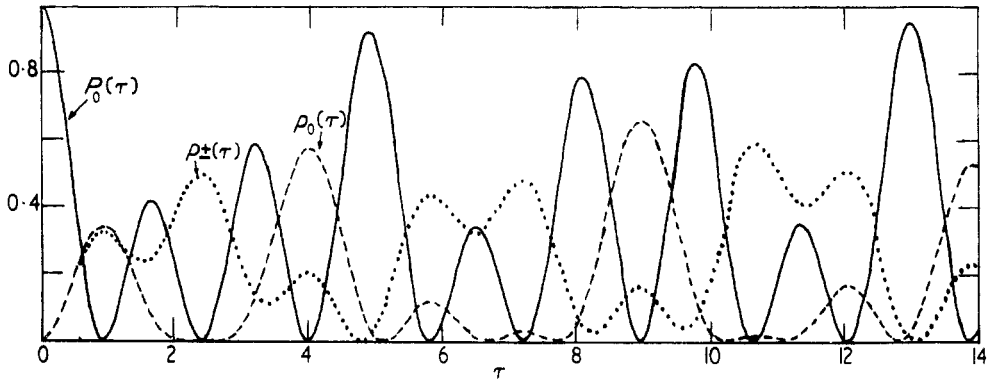

Figure 3. P_0, p_0 and $p_\pm$ as functions of τ in the three mode case for $\delta^2 = |g|^2$.

figure 2 shows clearly the almost periodic behaviour of $P_0(t)$. For the time range plotted here the energy never resides entirely in the mode at resonance, but at the time $|g|t \simeq 2{\cdot}4$, for example, practically all the energy resides in the two outlying field modes. At the times $|g|t \simeq 4{\cdot}9$ and $|g|t \simeq 13{\cdot}0$ the probability of the atom being in its excited state is a considerable fraction (greater than 0·9) of its original value. The events occurring at these times are the Poincaré recurrences, but they are not so sharply defined in this system as they are in a system with many degrees of freedom (see paper II).

In the limit of strong coupling, $\delta^2/|g|^2 \to 0$ the dominant terms in $P_0(t)$, $p_0(t)$ and $p_\pm(t)$ are

$$\left.\begin{aligned} P_0(t) &\sim \cos^2(\sqrt{3}|g|t) \\ p_0(t) &\sim \tfrac{1}{3}\sin^2(\sqrt{3}|g|t) \\ p_\pm(t) &\sim \tfrac{1}{3}\sin^2(\sqrt{3}|g|t) \end{aligned}\right\} \quad \delta^2 \ll |g|^2 \tag{62}$$

(Compare (50) where the argument of the trigonometric functions is $\sqrt{2}|g|t$.) In the limit of weak coupling, $\delta^2/|g|^2 \to \infty$, we find that the dominant terms are now

$$\left.\begin{aligned} P_0(t) &\sim \cos^2|g|t \\ p_0(t) &\sim \sin^2|g|t \\ p_\pm(t) &\sim 0 \end{aligned}\right\} \quad \delta^2 \gg |g|^2 \tag{63}$$

($p_\pm(t)$ has an amplitude of the order of $|g|^2/\delta^2$.) Equations (63) contrast strongly with equations (51); the different type of behaviour is clearly due to the presence of the mode on resonance.

As in the two mode case, a useful measure of the extent to which the energy is distributed over the atom and the field modes is given by the quantities $\bar{P}_0$, $\bar{p}_0$, and $\bar{p}_\pm$, which are easily calculated from (53), (54) and (55). We find

$$\bar{P}_0 = \frac{r_+^2 + r_-^2}{2x} = \frac{1}{2}\left(1 - \frac{4\Delta^2}{X}\right) \tag{64}$$

$$\bar{p}_0 = \frac{|g|^2}{2x}\left(\frac{r_-^2}{q_+^2} + \frac{r_+^2}{q_-^2}\right) = \frac{\Delta^4 + 3}{2X} \tag{65}$$

and

$$\bar{p}_{\pm} = \frac{|g|^2}{2x}(2\delta^2 + q_+^2 + q_-^2) = \frac{3(\Delta^2+1)}{2X} \tag{66}$$

where in the second half of these equations we have used (55), (56) and (61) for $q_{\pm}$, x, and $r_{\pm}$ to simplify the expressions and to write them in terms of the dimensionless variables

$$\Delta = \delta/|g| \qquad X = x/|g|^4 = \Delta^4 + 2\Delta^2 + 9. \tag{67}$$

These expressions give the average distribution of the total energy of the system over its constituent parts as a function of the mode separation measured in units of the coupling constant. Note that in the weak coupling limit ($\Delta \to \infty$) we have

$$\bar{P}_0 \to \bar{p}_0 \to \tfrac{1}{2} \qquad \bar{p}_{\pm} \to 0 \qquad (\Delta^2 \to \infty)$$

so that the energy tends to be distributed only over the atom and the mode on resonance, when the energy separation of the two outlying modes becomes large (as we might have expected). In the strong coupling limit ($\Delta \to 0$) we have

$$\bar{P}_0 \to \tfrac{1}{2} \qquad \bar{p}_0 \to \bar{p}_{\pm} \to \tfrac{1}{6}$$

so that in this case, as in the two mode case, the energy is divided in the first place equally between the atom and the field, and in the second place, equally amongst the three field modes. In the intermediate case, $\Delta^2 = 1$ (which we have plotted in figure 3) the values

$$\bar{P}_0 = \tfrac{1}{3} \qquad \bar{p}_0 = \tfrac{1}{6} \qquad \bar{p}_{\pm} = \tfrac{1}{4}$$

are obtained.

In figure 4 we have plotted $\bar{P}_0$, $\bar{p}_0$ and $\bar{p}_{\pm}$ as functions of Δ for the range $6 \geqslant \Delta \geqslant 0$. An interesting feature of the curves is that for values of $\Delta \leqslant 3$, more energy is concentrated in each of the outlying modes than in the mode on resonance, which is perhaps unexpected. Only in the extreme weak coupling limit is more energy concentrated in the resonant mode than in the outlying modes. We note that all three functions have turning points.

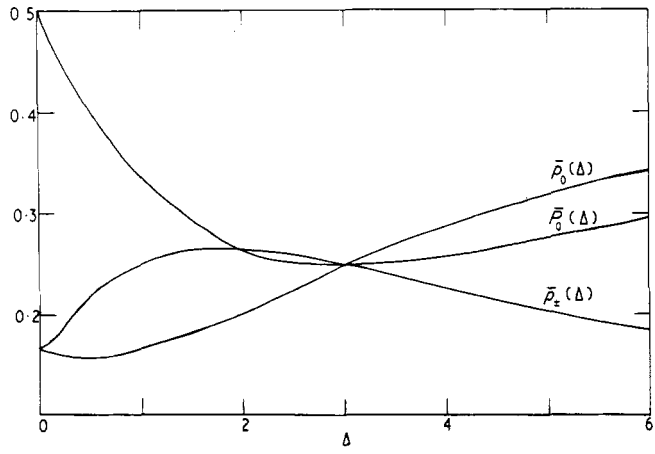

Figure 4. $\bar{P}_0$, $\bar{p}_0$ and $\bar{p}_{\pm}$ as functions of Δ in the three mode case.

As will be seen in paper II, the three mode case just discussed has many features in common with the N mode problem when N is large and there is a field mode on resonance with the atom. Examples are the almost periodic behaviour of $P_0(t)$, and the approximately sinusoidal behaviour in both the weak and strong coupling limits. It is therefore a useful analytically soluble model to study before tackling the more difficult N mode problem. Likewise the two mode problem has many features in common with the N mode problem in which there is no mode on resonance with the atom. However, for the particularly symmetric energy level scheme we have chosen in the two mode case, the almost periodic behaviour is lost, and $P_0(t)$ becomes truly periodic.

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